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Note

Parity and disparity subgraphs

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ABSTRACT

A parity subgraph of a graph is a spanning subgraph such that the degrees of each vertex have the same parity in both the subgraph and the original graph. Known results include that every graph has an odd number of minimal parity subgraphs. Define a disparity subgraph to be a spanning subgraph such that each vertex has degrees of opposite parities in the subgraph and the original graph. (Only graphs with all even-order components can have disparity subgraphs). Every even-order spanning tree contains both a unique parity subgraph and a unique disparity subgraph. Moreover, every minimal disparity subgraph is shown to be paired by sharing a spanning tree with an odd number of minimal parity subgraphs, and every minimal parity subgraph is similarly paired with either one or an even number of minimal disparity subgraphs.

Define a subgraph H of G to be a *parity subgraph* if H spans G and the degree of every vertex v has the same *parity* in both G and H (in other words, v either has odd degree in both or has even degree in both). Parity subgraphs were apparently introduced as such in [3]; the best reference is [15, Section 3.2]. Let $c(G) = |E(G)| - |V(G)| + \kappa$ denote the *cyclomatic number* of a graph G with κ components. It is known that every graph G has $2^{c(G)}$ parity subgraphs, that every spanning tree of G contains a parity subgraph of G, and that G always has an odd number of *minimal parity subgraphs* (parity subgraphs H of G that have no proper subgraph that is a parity subgraph of G).

Define a spanning subgraph H to be a *disparity subgraph* of G if the degree of every vertex v has opposite parity in G and H (in other words, the degree is odd in one and even in the other). Spanning subgraphs in general are often called 'factors'; indeed, some authors call parity subgraphs 'parity factors' to emphasize that they have to be spanning subgraphs. Disparity subgraphs could similarly be called 'disparity factors.' Fig. 1 illustrates parity and disparity subgraphs in a simple case.

We prove that every connected graph G of even order— which we note below are precisely the connected graphs that can have disparity subgraphs— has $2^{c(G)}$ disparity subgraphs and that every spanning tree of G contains a disparity subgraph of G (but the parity of the number of minimal disparity subgraphs is not determined). Indeed, we show that every spanning tree of such a G contains both a unique parity subgraph and a unique disparity of G. Furthermore, we prove that, for every minimal disparity subgraph H_d, there exists an odd number of minimal parity subgraphs H_p such that $H_d \cup H_p$ is acyclic, and that, for every minimal parity subgraph H_p, there exists either one or an even number of minimal disparity subgraphs H_d such that $H_d \cup H_p$ is acyclic.

1. Existence, dispersion, and quantity

For any graph, let $\mathcal{P}(G)$ and $\mathcal{D}(G)$ denote, respectively, the sets of all parity and disparity subgraphs of G. Since every connected graph has an even number of *odd vertices* (meaning vertices of odd degree), $\mathcal{D}(G) \neq \emptyset$ implies that every

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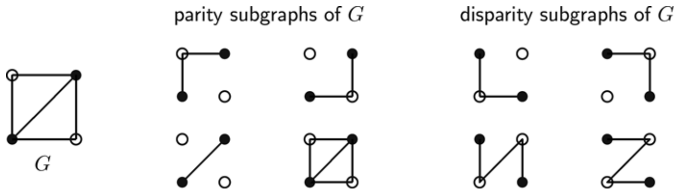

Fig. 1. A graph G with all its parity and disparity subgraphs (vertices of even degree are shown as 'hollow'; of odd degree, as 'solid').

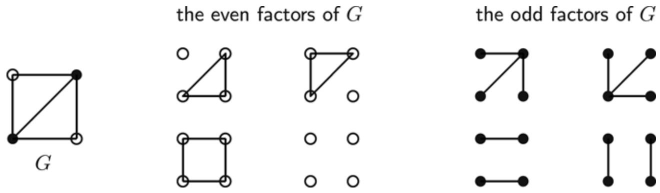

Fig. 2. The graph G from Fig. 1 with all its even and odd factors.

component of G also has an even number of *even vertices* (meaning vertices of even degree) and so that the *order* $|V(G)|$ of G is even. (Corollary 5 will show the converse for connected graphs.)

An *even* [respectively, *odd*] *factor* of G is a spanning subgraph H such that every vertex has even [odd] degree in H. The parity subgraph portion of Lemma 1 is Proposition 3.2.2(2) in [15].

Lemma 1. *Subgraph H of G is a parity [respectively, disparity] subgraph if and only if $E(G)-E(H)$ induces an even [odd] factor of G.*

Proof. This follows from the observation that inserting (or deleting) the edges of an even factor into a subgraph does not change the parity of any vertex degree, while inserting the edges of an odd factor reverses the parity of every vertex degree. □

Fig. 2 illustrates Lemma 1 for the graph in Fig. 1, with the even and odd factors shown in the same order that the corresponding parity and disparity subgraphs were shown in Fig. 1.

As is frequently done when discussing the cycle space of a graph G, let $H_1 \oplus H_2$ denote the symmetric difference operation (sometimes called the 'ring sum') of the edge sets of the subgraphs H_1 and H_2. For every even factor F, a subgraph H is a parity subgraph [or disparity subgraph or even factor or odd factor] if and only if $H \oplus F$ also is. Therefore, minimal parity subgraphs and minimal disparity subgraphs must be acyclic. Parity subgraphs – especially minimal parity sets – are frequently studied as 'postman sets' (or 'postman tours,' 'postman joins,' etc.), since their edges are the ones that need to be duplicated to make an arbitrary graph into an Eulerian multigraph; see [1,7,16] and [4, Section 5.4].

For any tree T with vertices u and v, let $T[u, v]$ denote the factor of T that contains the edges of the unique u-to-v path in T. Let $d_G(v)$ denote the degree of the vertex v in the graph G. Congruences will always be modulo 2. The proof of Theorem 2 parallels that of [2, Lemma 1].

Theorem 2. *Let T be a tree. If $U \subseteq V(T)$ is an arbitrary subset such that $|U|$ is even, then T has a unique factor H such that $d_H(u)$ is odd for each $u \in U$ and $d_H(v)$ is even for each $v \in V(T) - U$.*

Proof. If $U = \{u_1, u_2, \ldots, u_{2a}\}$, then

$$H = T[u_1, u_2] \oplus T[u_3, u_4] \oplus \cdots \oplus T[u_{2a-1}, u_{2a}]$$

is a factor such that $d_H(u)$ is odd for each $u \in U$ and $d_H(v)$ is even for each $v \in V(T) - U$.To show uniqueness, suppose that H_1 and H_2 are two such factors of T. Then $H_1 \oplus H_2$ is an even factor of T. Since T is acyclic, this even factor $H_1 \oplus H_2$ must be edgeless, and so $H_1 = H_2$. □

Corollary 3 (*Itai and Rodeh [9]*). *Every spanning tree of a connected graph G contains a unique parity subgraph of G.*

Corollary 4. *Every spanning tree of a connected even-order graph G contains a unique parity subgraph and a unique disparity subgraph of G.*

Corollary 5. *A connected graph G has a disparity subgraph if and only if G has even order.*

Proof. 'Only if' is immediate from graphs always having an even number of odd vertices. 'If' follows from Theorem 2 using any spanning tree of the connected graph G. □

For every connected graph G and $U \subseteq V(G)$ with $|U|$ even, let $\mathcal{A}_G(U)$ denote the set of all factors H of G such that $d_H(u)$ is odd for each $u \in U$ and $d_H(v)$ is even for each $v \in V(H) - U$.

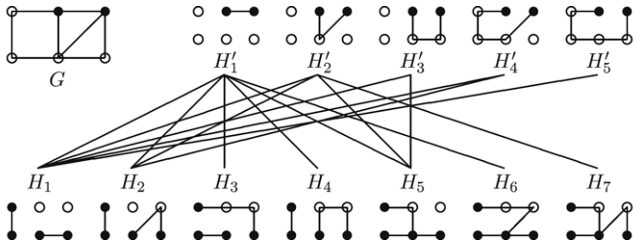

Fig. 3. A graph G with its minimal parity graphs $H'_1, \ldots, H'_5$ and its minimal disparity graphs $H_1, \ldots, H_7$, together with lines showing the 15 H_i, H'_j pairings such that $H_i \cup H'_j$ is acyclic.

Lemma 6. *Suppose G is a connected graph. If $H \in \mathcal{A}_G(U)$, then*

$$\mathcal{A}_G(U) = \{H \oplus F : F \text{ is an even factor of } G\}.$$

Proof. If $H \in \mathcal{A}_G(U)$, then automatically $H \oplus F \in \mathcal{A}_G(U)$ for each even factor F of G. Conversely, if $H, H' \in \mathcal{A}_G(U)$, then $H \oplus H' = F$ is an even factor of G, and so $H' = H \oplus F$.

The famous theorem of König [10] states that every graph G has exactly $2^{c(G)}$ even factors, and so G will have exactly $2^{c(G)}$ parity subgraphs by Lemma 1. Theorem 7 will show that every even-order graph has that same number of disparity subgraphs. This will also show that such graphs G have exactly $2^{c(G)}$ odd factors (giving a transparent proof of that last fact, which was originally proved in [11]; also see [5,6]). Theorem 7 follows from König's theorem, Theorem 2, and Lemma 6.

Theorem 7. *If G is a connected graph and $U \subseteq V(G)$ is an arbitrary subset with $|U|$ even, then $2^{c(G)} = |\mathcal{A}_G(U)|$.*

Suppose G is a connected graph with n vertices and m edges. There are 2^{n-1} ways to choose $U \subseteq V(G)$ with $|U|$ even, and G has $2^{c(G)}$ factors in $\mathcal{A}_G(U)$ by Theorem 7. This implies the well-known fact that G has exactly $2^{c(G)}2^{n-1} = 2^m$ factors.

Let G be a graph with κ components $G_1, G_2, \ldots, G_\kappa$. Using Theorem 8 and the fact that

$$2^{c(G)} = 2^{c(G_1)} \cdot 2^{c(G_2)} \cdots 2^{c(G_\kappa)},$$

we receive the next more general result.

Theorem 8. *Let G be a graph with κ components $G_1, G_2, \ldots, G_\kappa$, and let $U \subseteq V(G)$ with $|U \cap V(G_i)|$ even for all $i = 1, 2, \ldots, \kappa$. Then G has $2^{c(G)}$ factors H such that $d_H(u)$ is odd for each $u \in U$ and $d_H(v)$ is even for each $v \in V(H) - U$.*

The role of Lemma 6 in the proving that connected graphs have equal numbers of parity and disparity subgraphs can be viewed as giving a bijection between $\mathcal{P}(G)$ and $\mathcal{D}(G)$, with $E(G) - F \in \mathcal{P}(G)$ corresponding to $H \oplus F \in \mathcal{D}(G)$ (with $H \in \mathcal{D}(G)$ fixed and F ranging over all even factors), as well as a bijection between even factors F and odd factors $E(G) - (H \oplus F)$.

Lemma 9 is due to Hoffmann [8, Theorem 2]. It also follows from the elegant abstract approach of Wu in [14] (or from [12, Theorem 1] applied to the multigraph obtained by duplicating a set S of edges that constitute a postman set of G).

Lemma 9 (*Hoffmann [8]*). *Every (multi) graph has an odd number of maximal even factors.* □

Theorem 10 is a result for parity subgraphs that has no direct analogy for disparity subgraphs. It also follows from the matroidal approach of Woodall in [13]. (The multigraph portions of Lemma 9 and Theorem 10 will be used in the proofs of Theorems 12 and 14.)

Theorem 10. *Every connected (multi) graph has an odd number of minimal parity sub (multi) graphs.*

Proof. This follows from Lemmas 1 and 9. □

2. Pairing minimal parity and disparity subgraphs

Corollary 4 can be viewed as saying that every spanning tree T of a graph G pairs each minimal disparity subgraph H with a minimal parity subgraph H' such that $H \cup H'$ is in T; see Fig. 3. (Such a pairing can correspond to more than one spanning tree; in the example in Fig. 3, the acyclic graph $H_1 \cup H'_1$ is contained in seven spanning trees of G.)

Theorems 12 and 14 will show, respectively, that every minimal disparity subgraph is paired in this fundamental way with an odd number of minimal parity subgraphs, and that every minimal parity subgraph is paired in this way with either one or an even number of minimal disparity subgraphs. Lemmas 11 and 13 will provide machinery for the theorems' proofs. We will prove Lemma 11 and Theorem 12 in detail, each followed by an illustrative example (Figs. 4 and 5), and sketch the analogous proofs for Lemma 13 and Theorem 14.

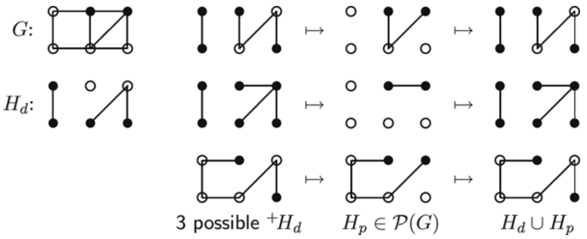

Fig. 4. An illustrative example for the proof of Lemma 11.

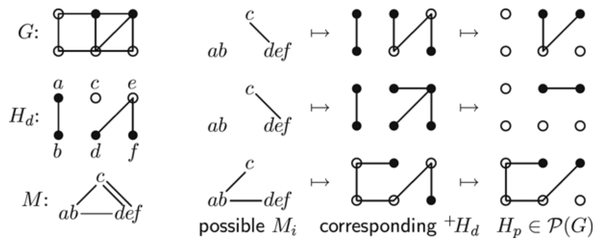

Fig. 5. Continuing the illustrative example in Fig. 4, for the proof of Theorem 12 (ab is the node for the component of H_d induced by $\{a, b\}$, etc.).

Lemma 11. *If H_d is a minimal disparity subgraph of G, then there is a bijection between the following two sets of subgraphs of G:*

(11a) *minimal subgraphs ^+H_d with $E(H_d) \subseteq E(^+H_d) \subseteq E(G)$ such that every component of ^+H_d has even order.*
(11b) *minimal subgraphs $H_p \in \mathcal{P}(G)$ such that $H_d \cup H_p$ is acyclic.*

Proof. Assume H_d is a minimal disparity subgraph of G (so $|V(G)|$ is even).

First suppose ^+H_d is as in 11a. Each ^+H_d is acyclic (by its minimality) and each component of ^+H_d contains an even number of vertices that are even vertices of G (corresponding to odd vertices of H_d). Since the components of ^+H_d have even order, each also contains an even set U of vertices that are odd vertices of G. By Theorem 2, ^+H_d contains a unique parity subgraph H_p of G as in 11b.

On the other hand, suppose H_p is as in 11b. Each component of $H_d \cup H_p$ contains an even number of vertices that are odd [respectively, even] vertices of G (corresponding to odd [even] vertices of H_p). Hence, each component of $H_d \cup H_p$ has even order and is as in 11a. □

Theorem 12. *For every minimal disparity subgraph H_d of G, there is an odd number of minimal parity subgraphs H_p of G such that $H_d \cup H_p$ is acyclic.*

Proof. Assume H_d is a minimal disparity subgraph of G (so $|V(G)|$ is even). Let M be the multigraph obtained by contracting the components of H_d into 'nodes' of M. Since $d_G(v) - d_{H_d}(v)$ is always odd, each node of M has even degree if and only if the corresponding component of G has even order. Theorem 10 implies that M will have an odd number of minimal parity submultigraphs $M_1, \ldots$, each corresponding to a subgraph ^+H_d as in 11a. Lemma 11 then ensures that those subgraphs ^+H_d are in one-to-one correspondence with the minimal parity subgraphs H_p of G as in 11b, and so that there will be an odd number of such subgraphs H_p. □

Lemma 13. *If H_p is a minimal parity subgraph of G, then there is a bijection between the following two sets of subgraphs of G:*

(13a) *minimal subgraphs ^+H_p with $E(H_p) \subseteq E(^+H_p) \subseteq E(G)$ such that every component of the forest ^+H_p has even order.*
(13b) *minimal subgraphs $H_d \in \mathcal{D}(G)$ such that $H_d \cup H_p$ is acyclic.*

The proof of Lemma 13 is analogous to the proof of Lemma 11.

Theorem 14. *For every minimal parity subgraph H_p of G, there is either one or an even number of minimal disparity subgraphs H_d of G such that $H_d \cup H_p$ is acyclic (with H_d unique if and only if every component of H_p has even order).*

Proof sketch. The proof of Theorem 14 is largely analogous to the proof of Theorem 12. We can suppose $|V(G)|$ is even (otherwise, the conclusion follows with zero H_d). The multigraph M obtained by contracting the components of H_p will be Eulerian (since $d_G(v) - d_{H_p}(v)$ is always even). Let U be the set of nodes of M that come from odd-order components of G, noting that $|U|$ is even since $|V(G)|$ is even. If $|U| = 0$, then H_d is unique by Corollary 4. Let $\mathcal{S}$ be the set of minimal submultigraphs M_i of M such that $U = \{v \in V(M) : \deg_{M_i}(v) \text{ is odd}\}$. Then there is a bijection between $\mathcal{S}$ and the subgraphs ^+H_p as in (13b).

We must show that $|\mathcal{S}|$ is even. One way to do this is to consider the incidence matrix of M augmented with two copies of the characteristic vector $\vec{u}$ of U. The augmented matrix constitutes an Eulerian matroid and so, by [13], $\vec{u}$ will be in an odd number of circuits of that matroid. One of these circuits will consist of the two $\vec{u}$ vectors, and the remaining even number of circuits that contain $\vec{u}$ will correspond to the members of $\mathcal{S}$. □

Corollary 15, a direct consequence of Theorem 14, is a partial analog to Theorem 10. (The examples shown in Figs. 1 and 3 show that the number of minimal disparity subgraphs can be either even or odd.)

Corollary 15. *Eulerian graphs have even numbers of minimal disparity subgraphs.*

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